CLASSIC MECHANICAL SPACE ON MANIFOLDS

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ABSTRACT

This paper explores the description of classic mechanical spaces within the framework of manifold theory, and introduces the applications of Lagrangian mechanics and Hamiltonian mechanics on differential manifolds. Initially, the concept and fundamental properties of differential manifolds are elucidated, emphasizing the significance of manifolds in mechanical problems. Subsequently, the paper discusses in details the representation methods of configuration space and velocity phase space on differential manifolds, and analyzes the evolution of Lagrangian mechanics and Hamiltonian mechanics in configuration space and phase space. In the discussion of Hamiltonian mechanics, this paper introduces the concepts of exterior algebra forms and differential forms, as well as the definition and characteristics of symplectic manifolds. Lastly, the paper explores the independence of generalized coordinates and generalized momenta in phase space, and points out the influence of dynamical equations on cotangent bundles and tangent bundles.

Keywords **classic mechanics · manifolds · configuration space · velocity phase space · phase space**

1 Introduction

When studying mechanical problems, it is crucial to understand the context (space) in which we are investigating the evolution of mechanical systems. We know that Lagrangian mechanics describes the motion of mechanical systems using configuration space, while Hamiltonian mechanics deals with the geometry of phase space, studying mechanical motion within phase space (of course, the application of phase space extends far beyond this). This article aims to take a higher perspective, using mathematical knowledge of differential manifolds to provide further explanation of the mechanical space, delve into why mechanical systems evolve in these spaces, and explain the issue of coordinate independence in mechanical spaces.

2 Lagrangian mechanic on manifolds

When considering the shape of space, is there a unified language to describe all spaces? For example, those of us on Earth may perceive the Earth's surface as two-dimensional, but when observed from space, it appears to be a two-dimensional sphere. How should we describe our world and other spaces? It is important to emphasize that the space in our physical reality is not flat and linear; it possesses a generally curved topological structure. Furthermore, when studying mechanical systems, we no longer confine our investigations to the three-dimensional Euclidean space. Instead, we use methods such as configuration space and phase space to gain a deeper understanding of mechanical systems. Therefore, it is necessary for us to understand the essence of "space," and manifold theory is used to describe it.

2.1 Differential manifolds

2.1.1 Definition

A set M with a finite chart, where each point can be represented in one chart, admits a differential manifold structure on M. For any point $\boldsymbol{p} \subset M$, there exists a neighborhood $U \subset M$ of x such that U is homeomorphic to an n-dimensional Euclidean space $\mathbb{R}^n$, then M is called an n-dimensional manifold. Alternatively, a manifold is a topological space locally equivalent to Euclidean space, which is a generalization of smooth surfaces.

An chart consists of open sets U in the Euclidean coordinate space $\boldsymbol{q} = (q_1, \cdots, q_n)$ and a one-to-one mapping $\varphi : U \to \varphi U \subset M$ onto some subset of M.

If two charts U and U' contain points $\boldsymbol{p}$ and $\boldsymbol{p}'$ in M with the same image, then $\boldsymbol{p}$ and $\boldsymbol{p}'$ each have neighborhoods $V \subset U$ and $V' \subset U'$ respectively, with the same image in M. This yields a mapping from a subset $V \subset U$ of one chart to a subset $V' \subset U'$ of another chart: $\varphi'^{-1}\varphi : V \to V'$.

This is a mapping from a region V in Euclidean space $\boldsymbol{q}$ to a region V' in Euclidean space $\boldsymbol{q'}$, given by n n-tuple functions: $\boldsymbol{q}' = \boldsymbol{q}'(\boldsymbol{q}), (\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{q}'))$. If $\boldsymbol{q}'(\boldsymbol{q})$ and $\boldsymbol{q}(\boldsymbol{q}')$ are both differentiable, these two charts are said to be compatible.

An atlas is the union of charts. A differential manifold is a class of equivalent atlases. Therefore, by using differential manifolds, we can extend a series of results from classical mathematical analysis, classical stochastic analysis, classical harmonic analysis, functional analysis, etc., to "non-linear spaces". By performing calculations locally equivalent to $\mathbb{R}^n$, manifolds allow for differential operations in "non-linear spaces". Moreover, every manifold can be embedded in some Euclidean space, and any open set in $\mathbb{R}^n$ is an n-dimensional manifold.[3]

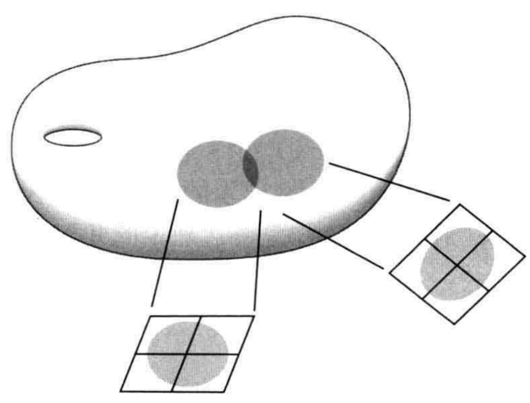

Figure 1: chart

2.1.2 Tangent space

Let M be a k-dimensional manifold embedded in E^n. At each point x, there exists a k-dimensional tangent space TM_x. Vectors in the tangent space TM_x with x as the starting point are called tangent vectors of M at x. The tangent vectors of M at x form a linear space TM_x, which is also known as the tangent space of M at x.

2.1.3 Tangent bundle

The union of tangent spaces $\bigcup_{x \in M} TM_x$ at each point of M has a natural differential manifold structure, with a dimension twice that of M. This manifold is called the tangent bundle of M, denoted as TM. Points in TM correspond to vectors tangent to M at some point x. Local coordinates of TM are given by: let $q_1, \cdots, q_n$ be local coordinates on M, and $\xi_1, \cdots, \xi_n$ be the components of the tangent vectors in this coordinate system, then $(q_1, \cdots, q_n; \xi_1, \cdots, \xi_n)$, which consist of $2n$ numbers, form a local coordinate system of TM.
It should be noted that using local coordinates to handle differential manifolds is analogous to considering a set

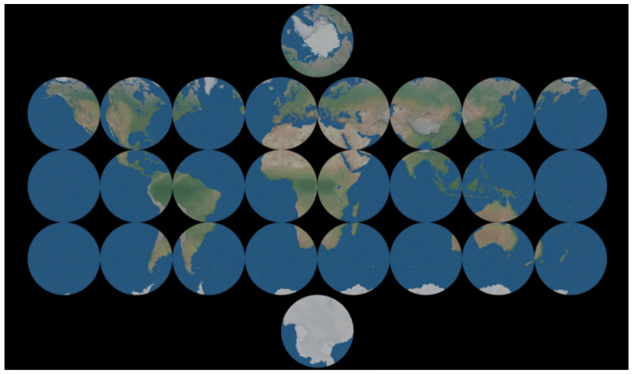

Figure 2: atlas

of basis vectors for linear spaces. In theoretical mechanics, using generalized coordinates is essentially studying mechanical systems using local coordinates on manifolds. Therefore, local coordinates are independent bases, and they determine the dimension of locally nested submanifolds.

The mapping $p : TM \to M$ that maps a tangent vector ξ to the tangent point $x \in U$ is called the natural projection. The pre-image $p^{-1}(x)$ of point $x \in M$ under the natural projection is the tangent space TM_x, which is called the fiber of the tangent bundle at point x.

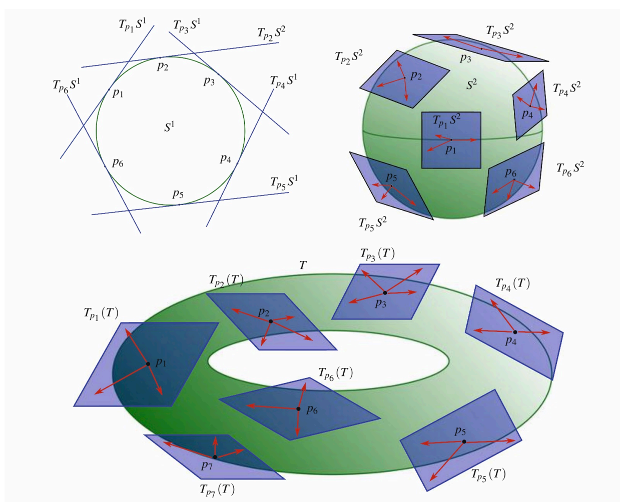

Figure 3: tangent bundle

2.2 Configuration space or velocity phase space

In Lagrangian mechanics, the motion of mechanical systems is described in configuration space, which has a differential manifold structure. For N particles, we can use a set of generalized coordinates $(q_1, q_2, \cdots, q_n)$ to form a set of local coordinates on the manifold. Let M be a differential manifold, TM be its tangent bundle, and $L : TM \to \mathbb{R}$ be a differentiable function. A mapping $\gamma : \mathbb{R} \to M$ is called a motion in a Lagrangian dynamical system with configuration manifold M and Lagrangian function $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ if it satisfies:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \frac{\partial L}{\partial \boldsymbol{q}}.$$

However, when we describe the evolution in configuration space under Lagrangian mechanics, rather than the evolution in velocity phase space, although the function $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ requires two variables to determine, during the evolution, the generalized coordinates and generalized velocities are not independent. There exists a constraint

$$\dot{q}_a = v_a = \frac{\mathrm{d}q_a}{\mathrm{d}t}$$

describing the evolution in configuration space rather than in velocity phase space. For the tangent bundle (i.e., velocity phase space) of the configuration space, the tangent vectors (generalized velocities) should be independent of the local coordinates (generalized coordinates). From a kinematic perspective, the generalized coordinates and generalized velocities at initial conditions do not depend on time. However, once we consider time evolution, the dimension of the tangent bundle is reduced by half, or the configuration manifold induces a determined trajectory $\varphi(t)$[4]. The constraint can be viewed as being imposed on a $n+1$ dimensional manifold by adding time t as a spatial dimension, such that the states of N particle systems at different times are embedded submanifolds in the $n+1$ dimensional manifold[2].

Therefore, in studying the evolution of mechanical systems, we assume that the manifold evolves over time, adding a time dimension. At this point, the generalized velocities are no longer independent, but are the time derivatives of the generalized coordinates. Hence, the Lagrangian function is written as $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$.

3 Hamiltonian mechanics

Hamiltonian mechanics is the geometry of phase space, which, as the cotangent bundle of configuration manifold, naturally possesses a symplectic structure, along with its Hamiltonian function. For each one-parameter canonical transformation in phase space, the symplectic structure remains invariant. For every one-parameter symplectic diffeomorphism group that preserves the Hamiltonian function, there exists a motion integral.

3.1 Exterior algebra form

In the previous section, it was mentioned that tangent vectors in the tangent space are essentially arrows with limited operational flexibility. However, from the perspective of duality, considering linear functions in the tangent space allows for more flexibility, as functions can be subjected to addition, multiplication, scalar multiplication, and composition operations. Once the concept of linear functions on the tangent space is introduced, one can further consider multilinear functions. Multilinear functions have multiple parameters, and they are linear with respect to each parameter.

Let $\mathbb{R}^n$ be an n-dimensional real vector space. A k-th exterior form (k-form) in the tangent space is a k-form skew-symmetric function defined on k vectors. Naturally, a 1-form is a linear function $\omega : \mathbb{R}^n \to \mathbb{R}$, which is the duality of vectors.

3.2 Differential form

For a manifold M, the k-differential form $\omega^k \mid_x$ at point x is the k-exterior form of M on the tangent space TM_x, to wit, a k-form skew- symmetric function defined on k vectors at point x where k vectors $\xi_1, \ldots, \xi_n$ tangent to M. For $\mathbb{R}^n$ space, considering $x_1, \ldots, x_n : \mathbb{R}^n \to \mathbb{R}$ as vector space $\mathbb{R}^n$ on the manifold M, fixing a point $\boldsymbol{x}$, there are n 1-forms $dx_1, \ldots, dx_n$ forming a basis for the 1-form space on the tangent space $T\mathbb{R}^n_{\boldsymbol{x}}$. Considering basic forms' wedge product:

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad i_1 < \cdots < i_k.$$

These C_n^k k-forms form a basis for the k-form space on $T\mathbb{R}^n_x$, so every k-form on $T\mathbb{R}^n_x$ can be expressed as:

$$\sum_{i_1 < \cdots < i_k} a_{i_1, \cdots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

3.3 Symplectic Manifold

3.3.1 Definition

For a $2n$-dimensional differentiable manifold M, if there exists a symplectic structure on M, namely a closed non-degenerate 2-differential form ω^2, such that $d\omega^2 = 0$ and $\forall \boldsymbol{\xi} \neq 0, \exists \boldsymbol{\eta} : \omega^2(\boldsymbol{\xi}, \boldsymbol{\eta}) \neq 0$ $\left(\boldsymbol{\xi}, \boldsymbol{\eta} \in TM^{2n}_{\boldsymbol{x}}\right)$, then $\left(M^{2n}, \omega^2\right)$ is called a symplectic manifold.

3.3.2 Cotangent Bundle and Symplectic Structure

For an n-dimensional manifold V, the 1-forms on TV_x are called cotangent vectors of V at x, and the cotangent vectors at x form an n-dimensional vector space, denoted as T^*V_x, the cotangent space. The union of all cotangent spaces on the manifold is the cotangent bundle, denoted as T^*V, which naturally has a $2n$-dimensional differentiable manifold structure. Considering q as a local coordinate system on V, the local coordinate system of TV can be given by $(q_1, \cdots, q_n; p_1, \cdots, p_n)$. And there exists a natural symplectic structure on the cotangent bundle. Proof is given in the appendix.

3.4 Phase Space

For the configuration manifold M, $\dot{q}$ is a tangent vector on M, and the generalized momentum $\boldsymbol{p} = \partial L/\partial \dot{\boldsymbol{q}}$ is a cotangent vector. Thus, the phase space formed by $\boldsymbol{p}, \boldsymbol{q}$ constitutes the cotangent bundle of M.

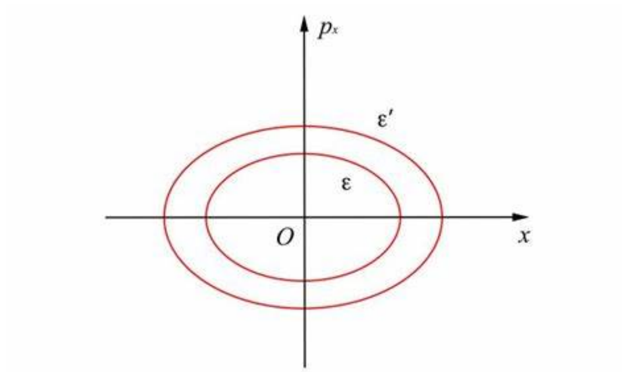

Figure 4: phase space

3.5 Independence of Generalized Coordinates and Momenta

Regarding phase space, i.e., the cotangent bundle, the independence of local coordinates indicates the independence of generalized coordinates and momenta. However, why do generalized coordinates and momenta remain independent due to time evolution in phase space? Time evolution does not directly affect the local coordinate basis on the cotangent bundle; compared to the tangent bundle, time evolution changes the dimension of the tangent bundle, influencing the independence between generalized velocities and generalized coordinates. Thus, studying mechanical systems on the tangent bundle (velocity phase space) is not a good choice. Here, we focus on the definition of generalized coordinates $\boldsymbol{p} = \partial L/\partial \dot{\boldsymbol{q}}$, naturally representing p as a 1-form (dual) of the tangent vector $\dot{q}$, derived from the Legendre transformation. This is determined by the initial conditions and remains part of the initial conditions in the subsequent dynamic process. As the dynamic equations are of second order, they do not directly constrain the first derivative (velocity), which is part of the initial conditions[1]. The phase space must contain information about the first derivatives, allowing us to construct independent quantities $\boldsymbol{p}$. Subsequent dynamic processes do not directly affect the cotangent bundle; they only affect the tangent bundle.

In the Hamiltonian formalism, we seek a curve in the cotangent bundle, which is the integral curve of the vector field , as the evolution of the initial state. This curve can be projected onto the base manifold to obtain a curve, but the reverse is not true; a curve in the base manifold cannot naturally lead to a curve in the cotangent bundle. Therefore, the dependence of **p** and **q** on the parameter **t** needs to be solved simultaneously. However, in the tangent bundle, the situation is reversed. A curve in the base manifold can naturally correspond to a curve in the tangent bundle, and the corresponding method is naturally to take the tangent vector of the curve. Therefore, the Lagrangian formalism essentially seeks a curve in the base manifold, where the tangent vectors of the curve are not independent of the curve itself.

The relationship between the tangent bundle and the cotangent bundle can be further discussed, but it goes beyond the scope of this discussion.

4 Conclusion

If we only focus on mathematics in $\mathbb{R}^n$, we will never use abstract mathematical concepts such as manifolds, tangent spaces, cotangent spaces, etc. However, it is these theories that provide us with powerful tools for studying the real world and mechanics, helping us solve various problems that classical mechanics cannot tackle. By understanding the mechanical space on manifolds, we can gain a deeper understanding of classical mechanics and better solve various problems in future studies.

A Proof: There exists a natural symplectic structure on the cotangent bundle

We first define a specific 1-form on T^*V. Let $\boldsymbol{\xi} \in T(T^*V)_p$ be a vector tangent to the cotangent bundle at $p \in T^*V_x$. The natural projection $f : T^*V \to V$ induces a differential $f_* : T(T^*V) \to TV$, which maps $\boldsymbol{\xi}$ to a vector tangent to V at x, denoted as $f_*\boldsymbol{\xi}$. Now, let's define the 1-form $\omega^1(\boldsymbol{\xi}) = \boldsymbol{p}(f_*\boldsymbol{\xi})$ on T^*V. In the given local coordinate system, $\omega^1 = \boldsymbol{p}d\boldsymbol{q}$. The closed form $\omega^2 = d\omega^1$ is non-degenerate, and the form ω^1 is referred to as the action.

The natural relationship between a tangent vector and a cotangent vector at the same point is pairing. For any locally defined function defining a cotangent vector, taking the directional derivative in the direction defined by any tangent vector yields a real number. This is a non-degenerate bilinear pairing, inducing the duality between the tangent space and cotangent space at that point, as well as between the tangent bundle and cotangent bundle. Since they are dual, for (paracompact) differential manifolds, the tangent bundle and cotangent bundle as real vector bundles are isomorphic. However, they are generally not canonically isomorphic, meaning that the specific way they are isomorphic may not be equivalent depending on the implementation.

$$
\begin{tikzcd}
T(T^*M) \arrow[r, "d\pi"] \arrow[d, "\pi_1"'] & TM \arrow[d, "\pi_2"] \\
T^*M \arrow[r, "\pi"] & M
\end{tikzcd}
$$

Figure 5: proof

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